Marking Systemic Portfolio Risk with Application to the Correlation Skew of Equity Baskets

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The downside risk of a portfolio of (equity)assets is generally substantially higher than the downside risk of its components. In particular in times of crises when assets tend to have high correlation, the understanding of this difference can be crucial in managing systemic risk of a portfolio. In this paper we generalize Merton's option formula in the presence jumps to the multi-asset case. It is shown how common jumps across assets provide an intuitive and powerful tool to describe systemic risk that is consistent with data. The methodology provides a new way to mark and risk-manage systemic risk of portfolios in a systematic way.

Introduction

It has been argued that one of the factors that triggered the downfall of Long Term Capital Management (LTCM) was its failure to incorporate "fat tails" of asset price distributions properly into investment decisions as well as risk management [1]. Today financial institutions systematically deduce fat tails from the option markets and incorporate this information consistently into the pricing as well as the risk management framework. Despite this progress it is interesting to note that little or even no efforts have been made in the marking and risk-managing of "fat tails" when individual assets are *combined* into a large portfolio. However a large part of a portfolio's fat tail is not the result of the fat tails of the individual components. It is this difference that can be associated with systemic risk of a portfolio that will generally dominate investment decisions in times of crises.

As pointed out [2] only roughly 50% of the downside risk (fat tail) of a portfolio of stocks such as the DAX, is due to the fat tails of its members. In a sense by ignoring the additional 50% of downside, financial markets have not yet fully learned the lessons from the LTCM debacle. In fact the current financial crisis is haunting us with the consequences of underestimating systemic risk in portfolio management.

In this paper we argue that financial institutions should extend their market data to account for systemic risk not only in the area of credit but across all assets classes whenever assets are combined into a portfolio. The proposed "extension of market data" would be similar, in some respect, to the generalization of at-the-money volatilities to all strikes.

The first objective of this paper is to show that the steepness of the DAX skew compared to its components can be linked to the systemic risk of the "DAX portfolio". This is important because it allows us to infer the systemic risk from the market consistently. The second objective of this paper is to explicitly construct a parameterization that describes the systemic risk of large equity portfolios. The latter can serve as a candidate for the extension of the market data mentioned above.

Recently, efforts have been made to extend Dupire's local volatility framework to incorporate state-dependent correlations into the dynamics of a portfolio [2, 3]. While this leads to a numerically efficient generalization of Dupire's model, there are two main drawbacks.

First, the model requires an a priori knowledge of the basket skew as an input. This is however not known in many cases. The method presented in this paper provides a natural way to interpolate skews to sub-baskets as well as cross-index baskets.

Second, while the extension of Dupire's model to local correlation consistently describes the correlation skew, it does not attempt to "explain" its causes. Consequently, it cannot be viewed as a straightforward methodology to monitor systemic risk. The approach presented in this paper defines a copula that explicitly links the correlation skew to systemic risk and hence could be a candidate to improve the (tail) risk-management of large portfolios. Work in this direction has been pursued by others [4],[8]. The paper is organized as follows: The first section reviews Merton's option formula that describes options in the presence of jumps. We then generalize this formula to the multi-asset case and show qualitatively how correlation skew can be induced in this framework. The next chapter generalizes the single-asset Merton formula in order to properly reproduce market prices of options at all strike prices. Finally we put both generalizations together and define the "Merton-Copula basket" that enables one to define systemic portfolio risk in a proper way.

A Merton basket option formula

In 1976, Merton generalized the Black-Scholes formula to situations where jumps are present [5]. In this case the dynamics of the stock price S_t is given by

$$\frac{dS_t}{S_t} = \left(r_t - q_t - \lambda E_\lambda \left[Y-1\right]\right) dt + \left(Y-1\right) dn_\lambda + \sigma dw_t \tag{1}$$

where $r_t, q_t, \lambda, dn_\lambda, \sigma, w_t$ denote the short rate, dividend yield, jump intensity, jump measure, diffusive vol and a Brownian motion respectively. The jump size $Y-1$ has a mean $\hat{k}$ and a lognormal volatility $+\delta$, i.e.

$$Y = \left(1+\hat{k}\right) e^{-\frac{1}{2}\delta^2 + \delta z}$$

where $z \in N\left(0,1\right)$. Equation 1 states that the dynamics of S_t is log-normal with occasional jumps occurring at rate λ and with size $Y-1$.

As the jumps are assumed proportional (rather than additive shifts) as well as fully uncorrelated to the diffusive part of the dynamics, the Merton price of a call option Call(T,K) with maturity T and strike K is simply given by an infinite sum of Black Scholes prices, i.e.

$$Call\left(T,K\right) = \sum_{n=0}^{\infty} e^{-\lambda T} \frac{\left(\lambda T\right)^n}{n!} BS\left(F\left(1+\hat{k}\right)^n e^{-\lambda \hat{k} T}, T, K, \sqrt{\sigma^2 + \frac{n\delta^2}{T}}\right) \tag{2}$$

where

$$BS\left(F,T,K,\sigma\right) = Df\left(F\,N\left(d_1\right) - K\,N\left(d_2\right)\right) \,:\, d_{1/2} = \frac{log\left(\frac{F}{K}\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{3}$$

denotes the standard Black-Scholes option formula.

It is very easy to interpret Eq.2 in intuitive terms: $e^{-\lambda T}\frac{(\lambda T)^n}{n!}$ is the probability of n jumps happening until maturity. In this case the spot has suffered an extra depreciation of $\left(1+\hat{k}\right)^n$ on average due to the jumps which is reflected in the adjustment to the forward price in the Black-Scholes formula.

The Merton model describes a simple way to incorporate volatility skew into the dynamics without loss of analytic tractability. However the model is too simple to give an exact fit to the observed volatility skew in practice. In Fig.1 we plot the implied volatility skew for different levels of the diffusive volatility σ. As σ increases the jump generates less skew which is consistent with intuition.

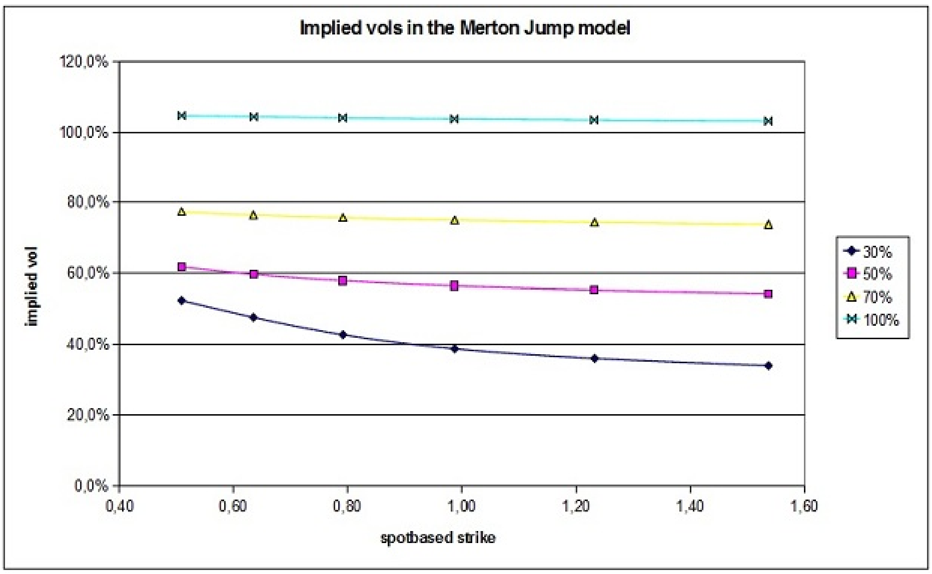

Fig.1: The volatility skew for different levels of diffusive vol. The effect of jumps on the skew becomes less and less pronounced for high values of the diffusive vol.

In the Appendix 1 we show that a way to retain a volatility skew for large σ stocks is to scale the jump size with the diffusive vol itself, e.g.

$$\hat{k} \to \left(\frac{\sigma}{\sigma_0}\right)^{\kappa} \hat{k} \tag{4}$$

σ_0, κ denotes a "vol-scale" and "vol-scale elasticity" respectively.

In order to generalize Merton's formula to basket options, Eq.1 can be generalized to N assets that evolve according to

$$
\begin{aligned}
\frac{dS_t^{(i)}}{S_t^{(i)}} &= \left(r_t - q_t^{(i)} - \lambda E_\lambda \left[Y^{(i)} - 1\right] dn_\lambda\right) + \sigma^{(i)} dw_t^{(i)} + \left(Y^{(i)} - 1\right) dn_\lambda, \\
\left\langle dw_t^{(i)}, dw_t^{(j)} \right\rangle &= \rho_{ij}^{diffusive} dt \qquad\qquad i = 1, \ldots, N
\end{aligned}
\tag{5}
$$

Note that, despite the multi-assets character of Eq.5 only a single Poisson process is "employed" just like in the case of equation 1. As we show later, this fact will prove crucial in order to generate correlation skew as well systemic risk of a portfolio.

Eq.5 states that the dynamics between jump events is governed by a multi-asset Black-Scholes dynamics with a diffusive correlation $\rho_{ij}^{diffusive}$

In the following we define a basket

$$
B_t = \sum_{i=1}^{n} \alpha_i S_t^{(i)} \tag{6}
$$

with fixed basket weights α_i. If Q denotes the pricing measure, Df the discount factor and $\hat{k}_i$ the jump size of asset i, then a call option on a basket can be calculated according to

$$
(Basket) - Call\,(T, K) = Df\, E^Q \left((B_t - K)^+\right) \tag{7}
$$

$$
= \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} TMPricer\left(F_1 \left(1 + \hat{k}_1\right)^n e^{-\lambda \hat{k_1} T}, \ldots, F_N \left(1 + \hat{k}_N\right)^n e^{-\lambda \hat{k}_N T}, \rho_{ij}^{diffusive}\right)
$$

Note that contingent to the number of jumps, the dynamics is log-normal and can hence well be approximated by a well known Black-Scholes three-moment basket pricer denoted by $TMPricer\,(F_1, \ldots, F_N, \rho)$(see for example [6]).

Equation 7 can be viewed as the generalization of Merton's formula to options on a basket of assets.

In Eq.5,7 all assets "share" the same Poisson process defined by $\left(\lambda, \hat{k}, \delta\right)$. This triple defines a Poisson process with intensity λ, a universal jump size $\hat{k}$ and jump volatility δ.

Individual jump-sizes $\hat{k}_i$ are computed from the "universal jump-size" by means of

$$
\hat{k}_i = \left(\frac{\sigma_i}{\sigma_0}\right)^{\kappa} \hat{k} \qquad\qquad i = 1, \ldots, N \tag{8}
$$

In the following we present results of Eq.7 in combination with Eq.8 when applied to the components of the DAX. Fig.2 shows the Merton basket implied vols that are inferred from Eq.7 versus the vol of the DAX for $\left(\lambda, \hat{k}, \delta, \sigma_0\right) = (25\%, -16\%, 18\%, 18\%)$

Table 1 shows how parameters change with the time horizon under consideration.

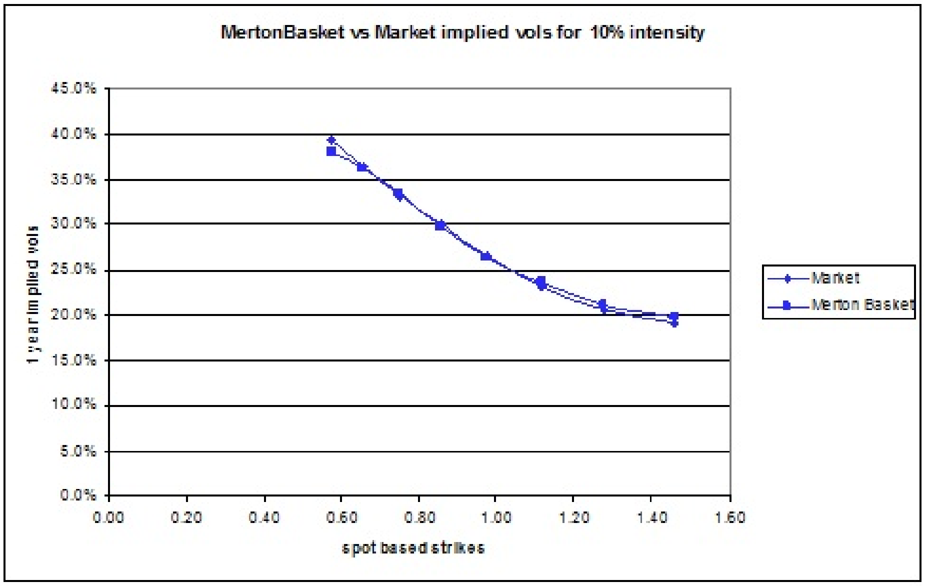

Fig.2: Reconstruction of DAX skew from its components: Merton basket implied vols versus market for $\left(\lambda, \hat{k}, \delta, \sigma_0\right) = (25\%, -16\%, 18\%, 18\%)$

Maturity	λ	$\hat{k}$	δk	σ_0
1 Year	0.25	-0.13	18%	0.12
2 Year	0.25	-0.137	15%	0.12
3 Year	0.25	-0.13	18%	0.127

Table 1: Jump parameter dependence as a function of time that is required to reconcile the skew of the DAX from its components. The parameters appear high relative to historic values. However they, in parts, reflect risk-premia that are associated with the sale of deep out-of-the-money put-options.

It is remarkable that the simple formulas of Eq.7,8 leads to quite good agreement with the skew of the DAX. We remind the reader that typically only 50% percent of the DAX skew can be explained in terms of the skew of the underlying assets [2]. Hence the idea of "sharing" a Poisson process across different assets

seems an idea worth investigating as a means to create copulas that are consistent with data. The intuition behind this concept is easily explained: When a far out-of-the money put on a basket ends up in the money at time of maturity, it is likely that at least one jump has occurred during the life of the option. As jumps occur simultaneously across all assets in our model, they effectively increase the implied correlation for out-of-the-money puts giving rise to correlation skew. The situation is reversed for out-of-the money call options that are likely to pay out only when no jump event has occurred. The implied correlation is expected to be close to the diffusive correlation in this case. For $\delta = 0$ the relationship between the diffusive correlation $\rho_{ij}^{diffusive}$ and total expected correlation ρ_{ij} can be calculated explicitly with the result

$$\rho_{ij} = \frac{\rho_{ij}^{diffusive} + \lambda \hat{k}_i \hat{k}_j}{\sqrt{\left(1 + \lambda \hat{k}_i^2\right)\left(1 + \lambda \hat{k}_j^2\right)}} \tag{9}$$

Note that in the limit of very frequent jumps where $\lambda \to \infty$ the correlation goes to one confirming our intuition. Eq. 9 plays a useful role when calibrating the set $\left(\lambda, \hat{k}, \delta, \sigma_0\right)$ to the skew of the DAX. This is because ρ_{ij} needs to be kept constant in order not to change the levels of the at-the-money basket vols for different choices of λ during calibration. Higher levels for the intensity require lower diffusive correlations which can be deduced from Eq.9.

Even though this explanation is highly plausible, it is not the only mechanism that yields tp correlation skew. For example, multi-asset stochastic volatility models that exhibit negative cross-correlations between spot returns of one asset with the vol of another asset generate cross-vanna sensitivity and hence correlations skew. It would be interesting to combine this effect with the one outlined above to a more complex model. The latter would decrease the intensities as well as the jumps-sizes of Table 1 that are required to achieve consistency between the index skew and the skew of its components. In many respects such a model could be considered the next step in the devlopment that is somewhat similar to progress in the modelling of FX derivatives where two different mechanism for the volatility skew- local vol and stochastic vol get combined to a more realistic description of this smarket.

We summarize this chapter by stating that the ideas presented seem worthwhile perusing. However the Merton models as well as the Merton basket formula need to be extended in order to make the calibration to the individual stock skews exact.

Single asset Merton extension

The main drawback of Merton's model is its failure to calibrate option prices exactly to the market. An example for the mis-calibration is given in Fig.3.

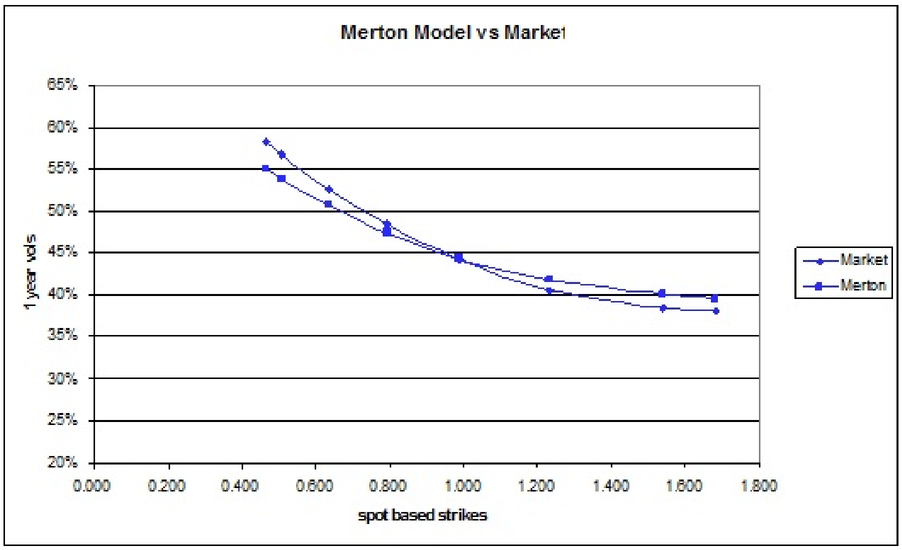

Fig.3:Example for mis-calibration of a single stock skew in the Merton model.

In order to fix this problem we generalize the Merton Single asset formula of Eq.2 to

$$Call\,(T,K) = \sum_{n=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\,BS\left(F\left(1+\lambda\hat{k}\right)^n e^{-\lambda\hat{k}T}, T, K, \sqrt{\sigma\,(K)^2+\frac{n\delta^2}{T}}\right) \tag{10}$$

The only difference is that the diffusive vol itself becomes a function of the strike price. The idea is that for a given set $\left(\lambda,\hat{k},\delta\right)$ the calibration mismatch is picked up by $\sigma\,(K)$.

The function $\sigma\,(K)$ can easily be determined by means of the fixed point algorithm [6]: The The algorithm starts with a constant zeroth guess σ for the diffusive volsurface $\sigma^{(0)}\,(K_j)=\sigma$ for M different values of strike K_j, $j=1,\ldots,M$. Iteration $i+1$ is obtained from iteration i according to $\sigma^{(i+1)}(K_j)=\sigma^{(i)}\,(K_j)+\Delta\sigma_j$where $\Delta\sigma_j$ is the vol-mismatch between the implied vol inferred from Eq.10 using $\sigma^{(i)}\,(K_j)$ and the market implied vol. For reasonable first guesses the algorithm typically converges very fast as the following diagrams show

Spotbased Strike	Diffusive Vol	Market Vol	Mispricing
0.51	40.0%	52.6%	1.2%
0.63	40.0%	48.4%	1.5%
0.79	40.0%	44.2%	0.7%
0.99	40.0%	40.6%	-1.0%
1.23	40.0%	38.4%	-2.7%
1.54	40.0%	38.2%	-2.8%

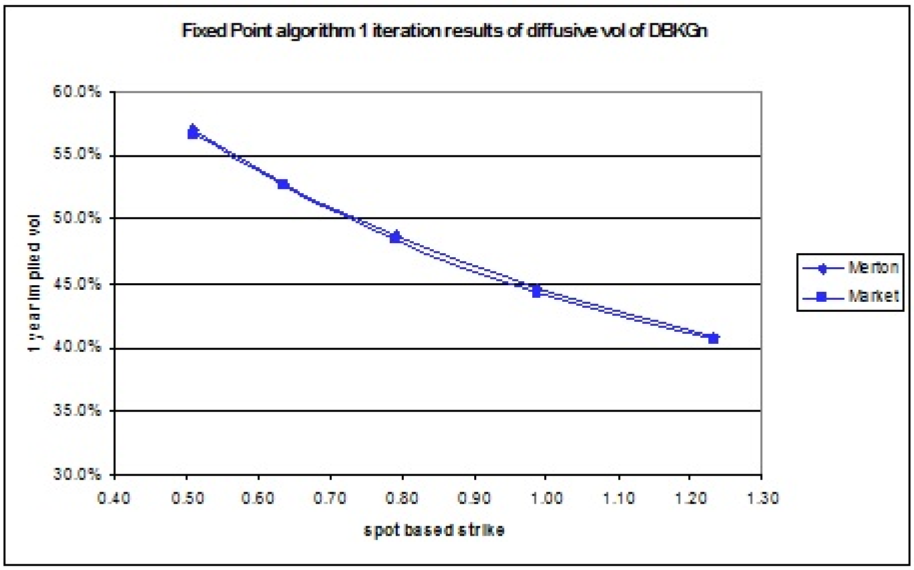

Fig.4: First iteration results for the fixed point algorithm:

Second iteration results for the fixed point algorithm:

Spotbased Strike	Diffusive Vol	Market Vol	Mispricing
0.51	40.9%	52.6%	0.1%
0.63	41.4%	48.4%	0.1%
0.79	40.4%	44.2%	0.1%
0.99	38.7%	40.6%	0.1%
1.23	37.1%	38.4%	0.1%
1.54	37.3%	38.2%	0.0%

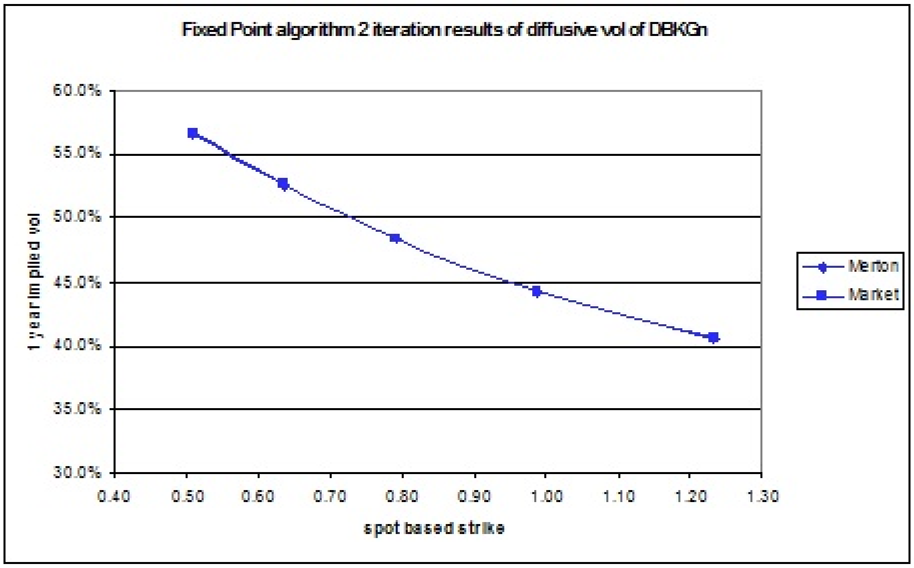

Fig.5: Second iteration results for the fixed point algorithm:

Equation 10 together with the fixed-point algorithm for the determination of the strike dependent diffusive skew defines the our "Merton single asset extension formula" for the pricing of options in the presents of jumps as well as "diffusive" contributions to the skew.

It is important to realize that one can easily construct a stochastic process explicitly that yields Eq.10 in the presence of "diffusive skew" $\sigma(K)$ for example:

$$\frac{dS_t}{S_t} = (r_t - q_t - \lambda E_\lambda [Y-1])\, dt + (Y-1)\, dn_\lambda + \hat{\sigma}(X_t, t)\, dw_t \qquad (11)$$

where $\hat{\sigma}(X_t, t)$ can be thought of a some kind of "local vol" with the difference that it is not a function of the spot S_t but rather the variable $X_t = \frac{S_t}{Y^n}$ where n is the number of jumps that have occurred up to time t.

The Merton Copula Basket

The goal of this chapter is to incorporate the results of the previous chapter into a basket options formula along the lines of Eq.7. However this is slightly more difficult as the three-moment approximation that was applied in Eq.7 is not appropriate in this case. This is because contingent to a specified number of jumps the dynamics is not log-normal anymore but given by a local volatility of Eq.11. In the following we outline the generalization of the three-moment pricer that was implicit in Eq.7 to "infinite" moments denoted by $BasketCopulaPricer\left(S_i, K, \rho, \sigma_i(K)\right)$

- **Step 1**: Dial a set of correlated normal deviates $w_i\, i = 1, \ldots, N$ from a correlation matrix ρ_{ij} and convert them into uniform deviates according to $u_j = N\left(w_j\right)$
- **Step 2**: Invert the individual asset (option implied) distributions according to
$$u_i = 1 + \frac{d}{dK}\frac{1}{Df}call\left(T, K\right)$$
to obtain a set $\left(S_T^{(1)}\left(u_1\right), S_T^{(2)}\left(u_2\right), \ldots, S_T^{(N)}\left(u_N\right)\right)$
- **Step 3**: Calculate $payoff = \left(\sum_{i=1}^{N} \alpha_i S_t^{(i)} - K\right)^+$
- **Step 4**: Go to step 1 Npath times, then average results and obtain

$$Call\left(T, K\right) \equiv BasketCopulaPricer\left(S_i, K, \rho, \sigma_i(K)\right)$$

We are now in a position to define the Merton Copula basket price in the presence of jumps as well as "local volatility" (see Eq.11)

$$Call\left(T, K\right) = \sum_{n=0}^{\infty} e^{-\lambda T}\frac{\left(\lambda T\right)^n}{n!} BasketCopulaPricer\left(S_i\left(1 + \hat{k}_i\right)^n e^{-\hat{k}_i T}, K, \rho_{ij}^{diffusive}, \sigma_i(K)\right) \quad (12)$$

where $\sigma_i(K)$ is given by the solution of the fixed point algorithm.

Fig.6 shows the results in the case of $\lambda = 0$ recovering the regular Gaussian Copula result. As mentioned in the beginning of the paper that not even 50% percent of the skew of the basket is recovered in this case from the skew of the individual assets.

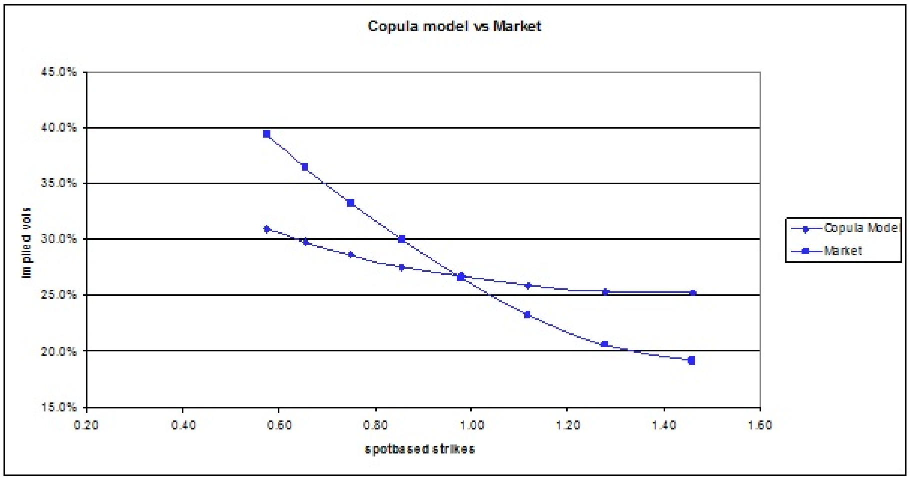

Fig.6: The Merton Copula basket reduces to the standard Gaussian Copula in the case of $\lambda = 0$. Less than 50% of the skew of the basket is recovered in this case.

Fig.7 shows the results after calibration. Note that for all choices of the calibration set $\left(\lambda, \hat{k}, \delta, \kappa\right)$ the approach is consistent with the skews of the individual assets.

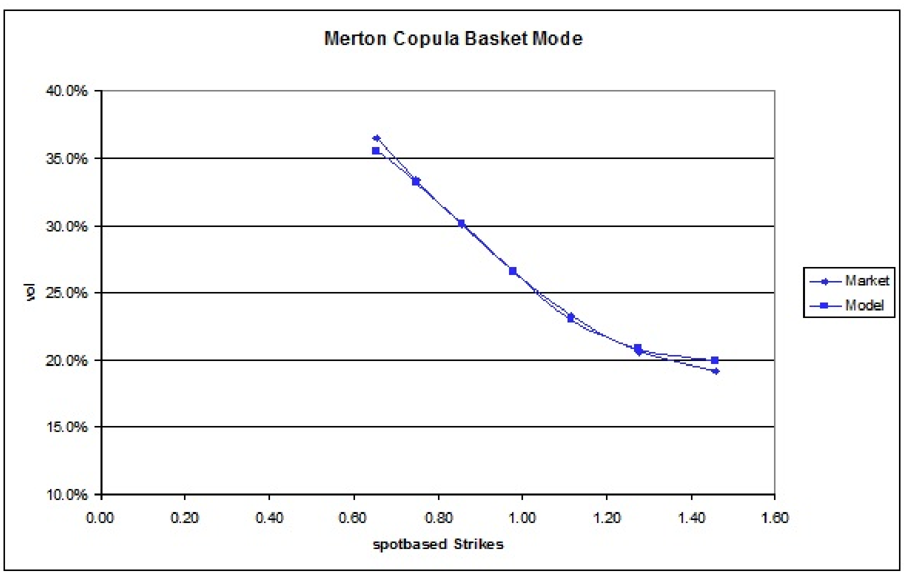

Fig.7: Calibration results of the DAX skew from its components for the set $\left(\lambda, \hat{k}, \delta, \kappa\right) = (0.35, -0.16, 18\%, 1, 17, 18\%)$.

Hence the set $\left(\lambda, \hat{k}, \delta, \kappa\right)$ describes "orthogonal new market-data" that quantifies the correlations skew. One expects a different set for each maturity.

Summary

We presented a generalization of the Merton Jump Option formula to multi-assets as well as diffusive skews and found that common jumps are an appropriate and intuitive way to define copulas that describe the systemic risk of a portfolio. Financial institutions should start treating systemic risk as "market data" across asset classes and start to mark, risk-manage and price these effects in a systematic way. For each maturity, a tuple $\left(\lambda, \hat{k}, \delta, \kappa\right)$ can serve as a candidate to describe this systemic risk in an appropriate way. It is straightforward to generalize our framework to more than one Poisson jump in the case different sources of systemic risk are encountered. This makes the framework well suited to mark skews of sub-baskets as well as cross-index baskets.

Appendix 1

In this Appendix we motivate the scaling of the jumpsize by the volatility σ according to Eq.4 for $\kappa = 1$ in order to retain the volatility skew for large σ.

From Eq.2 the price of a Digital-Put option of strike K and maturity T $Digital - Put(K,T) = E^{\beta}\left(1_{S_T<K}\right)$ is given by

$$Digital - Put\,(T,K) = \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} N(-d_2) \tag{13}$$

Hence the price change of the digital with a change in intensity λ is given by

$$\frac{\partial Digital - Put\,(T,K)}{\partial \lambda} = \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \varphi(-d_2) \frac{ln \frac{F(1+k)^n}{K}}{\sqrt{T}} \frac{k^2}{\sigma^2} \tag{14}$$

where $\varphi(x)$ denotes the normal density.This is due to the fact that for small values for λ the relationship between implied and diffusive vol can be approximated by $\sigma^2(implied) = \sigma^2(diffusive) + \lambda k^2$ and the last term in the numerator of Eq.3 is generally small for suffently out-of-the money options. Hence if one wants to retain the skew for large values of σ, the factor in Eq.14 suggests to scale the jump-size according to $k- > \sigma k$.

References

1. When Geniuses Failed: The Rise and Fall of Long-Term Capital, ISBN 0-375-75825-9

2. A dynamic model for correlation, RISK magazine Mar 2010, Alex Langnau

3. When Correlation breaks, RISK magazine Sep 2010, Adil Reghai
4. Alexander Lipton, Arthur Sepp, Credit value adjustment for credit default swaps via the structural default model, The Journal of Credit Risk (2009)
5. Continuous-Time Finance, R.C. Merton, Blackwell 1990
6. The hybrid most likely path, RISK April, Adil Reghai
7. Approximated moment-matching dynamics for basket options simulations, Brigo,Mercurio, Rapisarda, Scotti, 2001
8. Equity Correlations Implied by Index Options: Estimation and Model Uncertainty Analysis Rama Cont,Romain Dequest, April 19, 2010

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